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EFFECT OF VISCOSITY ON THE WAVE PROCESS IN A NONUNIFORM FLOW WITH A CRITICAL LEVEL

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A continuous analytical representation of an acoustic-gravitational field in a medium with a nonuniform flow in the presence of a critical layer is constructed. It is shown that taking into account the effect of viscosity eliminates singular values of the field.

A nonuniform flow (wind) forms a specific spatial structure of acoustic-gravitational waves [1–3]. In particular, a region with the so-called critical level can be formed. In the linear approximation without allowance for dissipation, both the velocity and density of the medium at this level turn to infinity. The energy of a perturbed field in an infinitely thin layer also becomes infinite. In this situation, a modification of the model adopted is required. One possible approach to "elimination" of infinities is based on making an allowance for dissipation. This approach raises the order of the system of equations and, as a result, there arises a "singularly perturbed problem" [4]. Weak dissipation brings about a small parameter, a coefficient at the higher derivative. This work is devoted to an analytical study of the spatial structure of an acoustic-gravitational wave under the above conditions.

The wave process is described by the following linear system of gas-dynamic equations taking into account weak dissipation:

$$\frac{d\rho}{dt} + \rho_0(z)\operatorname{div}\boldsymbol{v}' = 0, \qquad \rho_0(z)\frac{d\boldsymbol{v}}{dt} = \nabla P' - \rho' g \boldsymbol{e}_z + \eta\Delta\boldsymbol{v}' + \Big(\zeta + \frac{\eta}{3}\Big)\nabla\operatorname{div}\boldsymbol{v}',$$
$$\frac{dP}{dt} - a_0^2\frac{d\rho}{dt} = 0, \qquad P = P_0(z) + P', \qquad \rho = \rho_0(z) + \rho', \qquad \boldsymbol{v} = v_0(z)\boldsymbol{e}_x + \boldsymbol{v}'. \tag{1}$$

Here and below ρ, P, and $\boldsymbol{v}$ are the density, pressure, and velocity, η and ζ are the viscosities (assumed constant), g is the acceleration of gravity; x and z are the Cartesian coordinates; t is the time, $a_0 = (\gamma P_0 \rho_0^{-1})^{1/2}$ is the velocity of sound, and γ is the ratio of specific heats. The subscript 0 and the prime refer to the parameters of the medium in an unperturbed state and to their perturbations, respectively.

The unperturbed state of the medium at $\eta \neq 0$ is described by the relations

$$P_0(z) = P_0(0)\exp(-zH^{-1}), \qquad \rho_0(z) = \rho_0(0)\exp(-zH^{-1}), \qquad H = a_0^2 g^{-1}\gamma^{-1},$$
$$v_0(z) = w_0 z_0^{-1}(z - z_0), \qquad z_1 = z_0 w_0^{-1}(w_0 + \omega k^{-1}).$$

We study a two-dimensional acoustic-gravitational wave excited by a distribution of the vertical velocity of the medium at a level $z = \text{const}$ in the form of a stationary wave

$$v_z(t,x) = v_z(z)\exp(-i\omega t + ikx) \tag{2}$$

that propagates in the $\boldsymbol{x}$ direction with a velocity ωk^{-1}. Since the properties of the medium do not depend on the horizontal coordinate x, the perturbation in the $\boldsymbol{x}$ direction is also stationary. The total derivative d/dt in (1) can be represented as

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$$\frac{d\boldsymbol{f}}{dt} = ika_0 s(z)\boldsymbol{f}' + v_z' \frac{d\boldsymbol{f}_0}{dz}, \qquad s(z) \equiv a_0^{-1}[v_0(z) - \omega k^{-1}] = \delta_0 z_1^{-1}(z - z_1).$$

We consider the case with boundary condition (2) set at a certain level below z_1.

With allowance for viscosity ($\eta \neq 0$ and $\zeta \neq 0$), we represent the system of equations (1) in the form of four interrelated equations for v_x', v_z', ρ', and P':

$$\rho' = \frac{\rho_0}{a_0 s}\Big[\frac{i}{k}\Big(\frac{d}{dz} - \frac{1}{H}\Big)v_z' - v_x'\Big]; \tag{3}$$

$$v_x' - \frac{i\nu_1 s}{1 - s^2 + i\nu_2 s}\frac{d^2 v_x'}{dz^2} = \frac{i}{k(1 - s^2 + i\nu_2 s)}\Big[\frac{d}{dz} - \frac{1}{\gamma H} - \frac{s w_0}{a_0 z_0} + i\nu_3 s\frac{d}{dz}\Big]v_z'; \tag{4}$$

$$ika_0\rho_0 s v_z' = -\frac{dP'}{dz} - g\rho' + \Big(\zeta + \frac{\eta}{4}\Big)\frac{d^2 v_z'}{dz^2} - nk^2 v_z' + \Big(\zeta + \frac{\eta}{3}\Big)\frac{dv_x'}{dz}; \tag{5}$$

$$ika_0 s(P' - a_0^2\rho') - v_z'\frac{\gamma - 1}{\gamma}a_0^2\frac{d\rho_0}{dz} = 0. \tag{6}$$

Here ν_n ($n = 1$, 2, and 3) are dimensionless small parameters, $\nu_1 \equiv \eta k a_0^{-1}\rho_0^{-1}$, $\nu_2 \equiv (\zeta + 4\eta/3)ka_0^{-1}\rho_0^{-1}$, and $\nu_3 \equiv (\zeta + \eta/3)a_0^{-1}\rho_0^{-1}$. For $\nu_n = 0$, the order of system (3)–(6) is reduced, and the fields of v_x', v_z', ρ', and P' can be expressed in terms of a function $\Phi(z)$ that satisfies the differential equation

$$\frac{d^2\Phi}{dz^2} + D^2(z)\Phi = 0, \tag{7}$$

$$D^2(z) = \frac{\omega_1^2}{a_0^2 s^2} - k^2(1 - s^2) - \frac{1}{4H^2} - \frac{1}{Ha_0 s(1 - s^2)}\frac{2 - \gamma}{\gamma}\frac{w_0}{z_0} - \frac{3}{a_0^2(1 - s^2)^2}\frac{w_0^2}{z_0^2},$$

$$v_z' = \sqrt{1 - s^2}\,\exp(z/(2H))\Phi(z)\,\exp(-i\omega t + ikx),$$

$$v_x' = \frac{i}{k(1 - s^2)}\Big(\frac{d}{dz} - \frac{1}{\gamma H} - \frac{s w_0}{a_0 z_0}\Big)v_z', \qquad P' = -\frac{i\rho_0}{k(1 - s^2)}\Big[a_0 s\frac{d}{dz} - \frac{w_0}{z_0} - \frac{a_0 s}{\gamma H}\Big]v_z',$$

$$\rho' = \frac{\rho_0}{ika_0 s}\Big[\frac{s}{1 - s}\frac{d}{dz} + \frac{\gamma - 1 - \gamma s^2}{\gamma H(1 - s^2)} - \frac{s w_0}{a_0(1 - s^2)z_0}\Big]v_z', \qquad \omega_1^2 = \frac{(\gamma - 1)g^2}{a_0^2}.$$

For $z \to z_1$, we have $s(z) \to 0$. Ignoring dissipation, we obtain the following estimates for the fields: $P' \sim v_z' \sim (z - z_1)^{1-\alpha}$ and $\rho' \sim v_x' \sim (z - z_1)^{-\alpha}$. Here $\alpha \equiv [1 + (1 - 4R_i)^{1/2}]/2$ and $R_i \equiv 4((\gamma - 1)/\gamma)(a_0^2/H^2)(z_0^2/w_0^2)$. If $4R_i < 1$, then $v_x' \to \infty$ and $\rho' \to \infty$, and for $z \to z_1$, the conditions for linearization of the system of gas-dynamic equations are violated. The layer in the vicinity of $z = z_1$ is called the critical layer.

We divide the z axis into five zones (Fig. 1). In zones 1 ($z \ll z_1$) and 2 ($z \gg z_1$), we construct an "external" representation of the field based on approximation (7) [below, the factor $\exp(-i\omega t + ikx)$ is omitted]:

$$v_z^{(1)} \approx A^{(1)}\sqrt{1 - s^2}\exp\Big(\frac{z}{2H}\Big)\Phi^{(1)}(z), \qquad v_z^{(2)} \approx A^{(2)}\sqrt{1 - s^2}\exp\Big(\frac{z}{2H}\Big)\Phi^{(1)}(z).$$

Here $\Phi^{(1)}(z)$ is a solution of Eq. (6) that satisfies the condition

$$\Phi^{(1)} \sim \exp\Big(-i\int\limits_z D(z')\,dz'\Big)$$

for $z \to \infty$ and $\operatorname{Re} D > 0$.

The coefficient $A^{(1)}$ is found from boundary condition (2), and $A^{(2)}$ is determined below from the condition of matching of the fields in neighboring zones. In zone 3 ($z \approx z_1$) we construct, on the basis of

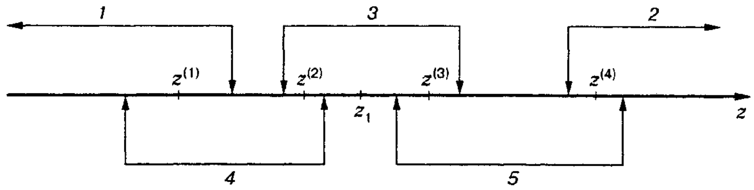

Fig. 1

system (3)–(6) with $\nu_n \ll 1$ and $|s(z)| \ll 1$, an "internal" representation of the fields, which possesses the property of finiteness.

Provided that the condition

$$\left|\frac{\rho'}{\rho_0}\right| \gg \left|\frac{v'_z}{sa_0}\right| \frac{\gamma - 1}{\gamma k H} \tag{8}$$

is valid for $z \to z_1$, we have $P' \approx a_0^2 \rho'$ according to (6). We represent Eq. (5) in the form

$$\frac{d\rho'}{dz} + \frac{\rho'}{\gamma H} \approx q_1(z, \nu_n),$$

where

$$q_1(z,\nu_n) \equiv -ika_0^{-1}\rho_0 s v'_z + \left(\zeta + \frac{4\eta}{3}\right) a_0^{-2} \frac{d^2 v'_z}{dz^2} - \eta a_0^{-2} k^2 v'_z + ika_0^{-2}\left(\zeta + \frac{\eta}{3}\right)\frac{dv'_x}{dz}.$$

If

$$|q_1| \ll |\rho' \gamma^{-1} H^{-1}|, \tag{9}$$

we obtain

$$\rho' \approx A^{(3)} \exp\left(-\frac{z - z_1}{\gamma H}\right); \tag{10}$$

$$P' \approx A^{(3)} a_0^{-2} \exp\left(-\frac{z - z_1}{\gamma H}\right). \tag{11}$$

For $z \to z_1$, Eq. (4) reduces to

$$\frac{i}{k}\frac{dv'_z}{dz} - v'_x \approx -\frac{i\nu_1 s}{k^2}\frac{d^2 v'_x}{dz'^2} + q_2, \tag{12}$$

where $q_2 \equiv -\frac{i}{k}\left[-\frac{1}{\gamma H} - \frac{sw_0}{a_0 z_0} + i\nu_3 s \frac{d}{dz}\right] v'_z$.

We restrict our consideration to the case in which the condition

$$|q_2| \ll \left|\frac{\nu_1 s}{k^2}\frac{d^2 v'_x}{dz^2}\right| \tag{13}$$

is valid. In view of (12) and (13), we find from (3)

$$\rho' \approx \frac{\rho_0}{a_0 s}\left[-\frac{i\nu_1 s}{k^2}\frac{d^2 v'_x}{dz^2} - \frac{i}{kH} v'_z\right], \tag{14}$$

assuming additionally that

$$\left|\frac{v'_z}{kHs}\right| \ll \left|\frac{v'_z}{k^2}\frac{d^2 v'_x}{dz^2}\right|. \tag{15}$$

According to (10), (14), and (15), we have

$$\frac{d^2 v'_x}{dz^2} \approx ia_0 k^2 \nu_1^{-1} A^{(3)} \rho_0^{-1}(0) \exp\left(\frac{z}{H} - \frac{z - z_1}{\gamma H}\right),$$

and, for $z \to z_1$,

$$v'_x \approx i a_0 k^2 A^{(3)} [2\nu_1 \rho_0(z_1)]^{-1}(z - z_1)^2. \tag{16}$$

By virtue of (12), we have

$$v'_z \approx i k a_0 \delta_0 [2 z_1 \rho_0(z_1)]^{-1} A^{(3)} (z - z_1)^2. \tag{17}$$

Conditions (8), (9), (13), and (15) limit the area of applicability of the "internal" representation (10), (11), (16), and (17): $|z - z_1| \ll 2H$. The relation $v'_x \sim \nu_1^{-1}$ is valid, where $\nu_1 \ll 1$ and $|v'_x/v'_z| \approx |kz_1/(\nu_1\delta_0)|$. For $z \to z_1$ and $\nu_1 \neq 0$, there is no singularity in the fields of v'_x and ρ'. Thus, a second undetermined parameter $A^{(3)}$ appears.

To study the fields in "intermediate" zones 4 and 5, we introduce a new dimensionless variable $y \equiv (z - z_1)/\mu(\nu_1)$ such that $\mu(\nu_1) \to 0$ as $\nu_1 \to 0$; $y \approx 1$; $\mu\nu_1^{-1} \to \infty$.

The field $f(y, \mu)$ can be expanded as

$$f(y,\mu) \approx \mu^{-1} f_{-1}(y) + f_0(y) + \mu f_1(y) + \ldots, \qquad \mu(\nu_1) \to 0, \qquad \nu_1 \to 0.$$

Equation (6), in view of (3) and (4), becomes

$$P' = \frac{\eta}{ik\mu^2}\frac{d^2 v'_x}{dy^2} + \left(\zeta + \frac{\eta}{3}\right)\mu^{-1}\frac{dv'_z}{dy} - a_0\rho_0\left(\mu\frac{\delta_0}{z_1}y - i\nu_2\right)v'_x - \frac{\rho_0 w_0}{ikz_0}v'_z. \tag{18}$$

Equation (4) relates $v'_x(y,\mu)$ and $v'_z(y,\mu)$:

$$v'_x - \frac{i}{k\mu}\frac{dv'_z}{dy} = -\frac{i}{\gamma k H}v'_z[1 + O(\mu)]. \tag{19}$$

With account of (3) and (19), we obtain the relation

$$\rho' = -\frac{i\rho_0(z_1)z_1(\gamma - 1)}{a_0\delta_0 k H \gamma \mu y}v'_z[1 + O(\mu)]. \tag{20}$$

From (18) and (19), we find the dependence $P'(v'_z)$:

$$P' = \left[\frac{\eta}{\mu^3}k^{-2}\frac{d^3 v'_z}{dy^3} - \frac{i a_0 \rho_0(z_1)\delta_0}{kz_1}y\frac{dv'_z}{dy} + \frac{i\rho_0(z_1)w_0}{kz_0}v'_z\right][1 + O(\mu)]. \tag{21}$$

Thus, we obtained relations (19)–(21) that permit determination of v'_x, ρ', and P' with accuracy to $O(\mu)$, provided that the function v'_z is known. Equation (6) reduces to

$$\frac{dP'}{dy} = g\mu\rho'[1 + O(\mu)], \tag{22}$$

and in view of (20)–(22), we have a fourth-order equation for $v'_z(y,\mu)$. For $\mu(\nu_1) = z_m \nu_1^{1/3}$ ($m = 4$ for zone 4 and $m = 5$ for zone 5), the function v'_z is independent of the parameter μ:

$$\frac{d^4 v'_z(y)}{dy^4} - iB_1^{(m)} y \frac{d^2 v'_z(y)}{dy^2} - iB_2^{(m)}\frac{dv'_z(y)}{dy} - iB_3^{(m)} y^{-1} v'_z(y) \approx 0. \tag{23}$$

Here $B_1^{(m)} = \delta_0 k^2 z_m^3 z_1^{-1}$, $B_2^{(m)} = B_1^{(m)}(1 - \delta_0^2)$, and $B_3^{(m)} = (\gamma - 1)k^2 z_m^3 z_1/(\gamma^2 \delta_0 H^2)$.

In zones 4 and 5, we have

$$(v'_z)^{(m)} \approx \sum_{n=1}^{4} C_n^{(m)} F_n^{(m)}(y), \tag{24}$$

where $F_n^{(m)}$ ($n = 1, 2, 3$, and 4; $m = 4$ and 5) are the linearly independent solutions of Eq. (23).

Expression (24) includes 10 arbitrary parameters $C_n^{(m)}$ and z_m ($n = 1, 2, 3$, and 4; $m = 4$ and 5). The intermediate representations in zones 4 and 5 are

$$v'_z \approx \varphi_0(y) + \mu\varphi_1(y) + \mu^2\varphi_2(y) + \ldots,$$

$$v'_x \approx \mu^{-1}\psi_{-1}(y) + \psi_0(y) + \mu\psi_1(y) + \ldots, \qquad \psi_{-1} = -\frac{i}{k}\frac{d\varphi_0}{dy},$$

$$\rho' \approx \mu^{-1} f_{-1}(y) + f_0(y) + \mu f_1(y) + \ldots, \qquad f_{-1} = -\frac{i\rho_0(z_1)kz_1(\gamma-1)}{a_0\delta_0 kH\gamma}\,\frac{\varphi_0(y)}{y},$$

$$P' \approx \Phi_0(y) + \mu\Phi_1(y) + \mu^2\Phi_2(y) + \ldots, \qquad \mu(\nu_1) \to 0, \qquad \nu_1 \to 0,$$

$$\Phi_0 = \frac{a_0\rho_0(z_1)}{k^3 z_m^3}\,\frac{d^3\varphi_0}{dy^3} - \frac{i\mu\rho_0(z_1)\delta_0}{kz_1}\,y\,\frac{d\varphi_0}{dy} + \frac{i\rho_0(z_1)w_0}{kz_0}\,\varphi_0.$$

Matching of the fields of v_x', v_z', ρ' and P' is performed simultaneously at four levels, $z^{(1)}$, $z^{(2)}$, $z^{(3)}$, and $z^{(4)}$ (see Fig. 1), with allowance for the first terms of the series, φ_0, ψ_{-1}, f_{-1}, and Φ_0. Here, we have 16 equation in 16 parameters: $z^{(n)}$ $(n = 1, \ldots, 4)$, $z^{(m)}$ $(m = 4, 5)$, $C_n^{(m)}$, $A^{(2)}$, and $A^{(3)}$. Derivatives can be discontinuous at matching points.

Thus, making allowance for dissipation leads to elimination of infinite values of the fields of v_x' and ρ' at $z = z_1$ [see (14), (16), and (17)]. The use of "internal" and "external" series allows a continuous representation of the fields of v_x', v_z'. ρ', and P' to be constructed.

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